Parameters estimation via unrolling for the PNPD algorithm

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Joint work with
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July 2, 2026

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MMIPU 2026 workshop
ICCSA 2026, University of Minho, Braga, Portugal

Outline

PNPD algorithm

Linear inverse problem

Consider a linear inverse problem of the form

$$b^\delta = Au + \eta_\delta = b + \eta_\delta, \tag{1}$$

where

- $b^\delta \in \mathbb{R}^s$ is the observed data corrupted by noise $\eta_\delta \in \mathbb{R}^s$
- $A \in \mathbb{R}^{s \times d}$ is a linear operator
- $u \in \mathbb{R}^d$ is the unknown signal we want to recover

Optimization problem

A common variational approach to solve eq. (1) is to consider the following optimization problem

$$\begin{aligned} &\underset{u\in\mathbb{R}^d}{\operatorname{argmin}} \ \frac{1}{2}\|Au - b^\delta\|^2 + h(Wu) \\ =\ &\underset{u\in\mathbb{R}^d}{\operatorname{argmin}} \ f(u) + h(Wu), \end{aligned} \tag{2}$$

where

- $\|.\|$ denotes the Euclidean norm
- $f(u)$ is the data fidelity term
- $h(Wu)$ is the regularization term
- $W \in \mathbb{R}^{d' \times d}$ is a linear operator
- $h : \mathbb{R}^{d'} \to \mathbb{R} \cup \{\infty\}$ is a convex and possibly non-smooth function.

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- $\operatorname{prox}_g(v) = \operatorname{argmin}_u \ \frac{1}{2}\|u - v\|^2 + g(u)$ is the proximal operator of g.

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Equation (3) assumes that the proximal operator of $h \circ W$ can be computed in closed form. This is not the case for some common regularization terms, such as the Total Variation (TV).

Nested Primal-Dual (NPD)

The Nested Primal-Dual (NPD)[1][2] algorithm is an inexact proximal gradient method summarized as

$$\begin{cases} \bar{u}_n = u_n + \gamma_n(u_n - u_{n-1}) & \leftarrow \text{Nesterov extrap.} \\ u_{n+\frac{1}{2}} = \bar{u}_n - \alpha_n \nabla f(\bar{u}_n) & \leftarrow \text{gradient descent} \\ u_{n+1} \approx \operatorname{prox}_{\alpha_n h \circ W}(u_{n+\frac{1}{2}}) & \leftarrow \text{inexact proximal step} \end{cases}$$

where $\gamma_n \geq 0$ is the extrapolation parameter like in FISTA.

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The inexact proximal step is computed by $k_{max} \in \mathbb{N}$ steps of a dual sequence v^k:
Let $u \in \mathbb{R}^d$, $v^0 \in \mathbb{R}^{d'}$, $\alpha > 0$, $0 < \beta < 2/\|W\|^2$, and define the sequence

$$v^{k+1} = \operatorname{prox}_{\beta\alpha^{-1} h^*}(v^k + \beta\alpha^{-1} W(u - \alpha W^T v^k)),$$

having limit $\lim_{k\to\infty} v^k = \hat{v}$. Then we have

$$\operatorname{prox}_{\alpha h \circ W}(u) = u - \alpha W^T \hat{v},$$

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Preconditioned NPD

Applying left preconditioning to the least squares regularized problem, the norm of the data fidelity term changes. Given S positive definite, we have

$$\operatorname*{argmin}_{u \in \mathbb{R}^d} \frac{1}{2} \|Au - b^\delta\|^2_{S^{-1}} + h(Wu), \tag{4}$$

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$$f_S(u) = \|Au - b^\delta\|^2_{S^{-1}} = \|S^{-\frac{1}{2}}(Au - b^\delta)\|^2.$$

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then NPD applied to eq. (4) gives the Preconditioned NPD (PNPD) algorithm:

$$\begin{cases} \bar{u}_n = u_n + \gamma_n(u_n - u_{n-1}), \\ u_{n+\frac{1}{2}} = \bar{u}_n - \alpha_n P^{-1}\nabla f(\bar{u}_n), \\ u_{n+1} \approx \operatorname{prox}_{\alpha_n h\circ W}(u_{n+\frac{1}{2}}), \end{cases}$$

Preconditioner choice

To satisfy the condition in eq. (5), we can choose S and P as follows:

$$P = A^T A + \nu I, \qquad S = AA^T + \nu I,$$

with $\nu > 0$. With this choice, the gradient step is equivalent to the Levenberg-Marquardt method applied to linear inverse problems, or equivalently, to the Iterated Tikhonov method[3].

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In general, the identity in eq. (5) is satisfied whenever P is a polynomial of $A^T A$ and S is a corresponding polynomial of AA^T.

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Unrolling architecture

Unrolling (single step)

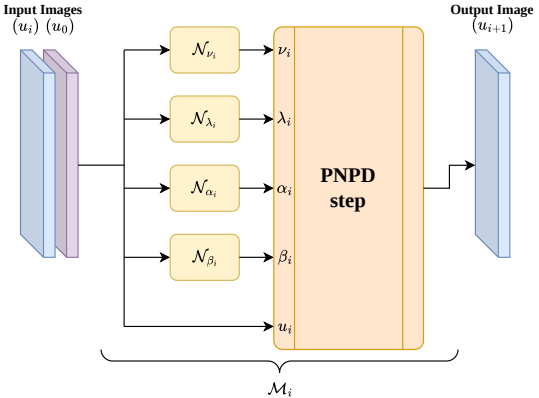

Figure 1: Single iteration of the unrolling architecture $\mathcal{M}_i$ composed by the predictors for each parameter $\mathcal{N}_{\nu_i}, \mathcal{N}_{\lambda_i}, \mathcal{N}_{\alpha_i}, \mathcal{N}_{\beta_i}$ and by one step of the PNPD algorithm.

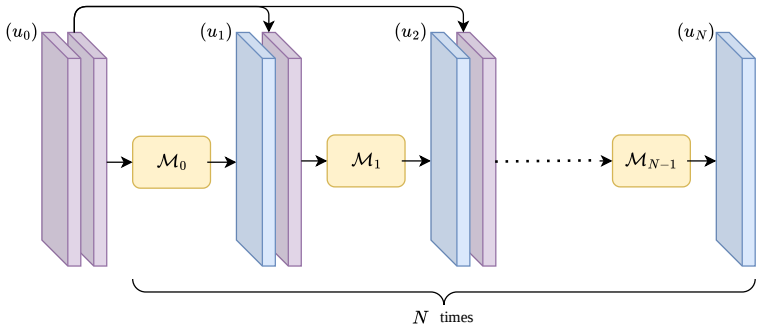

Figure 2: Global unrolling architecture obtained by composing the base network $\mathcal{M}_i$ N times.

Training

Training loss

The training loss is defined as a weighted average of a base loss function ℓ computed on each intermediate iterate

$$\mathcal{L}(u_0, \dots, u_{N-1}, u) = \frac{\sum_{i=0}^{N-1} w_i \, \ell(u_i, u)}{\sum_{i=0}^{N-1} w_i}.$$

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and we choose the weights w_i as

$$w_i = \frac{1}{N - i}.$$

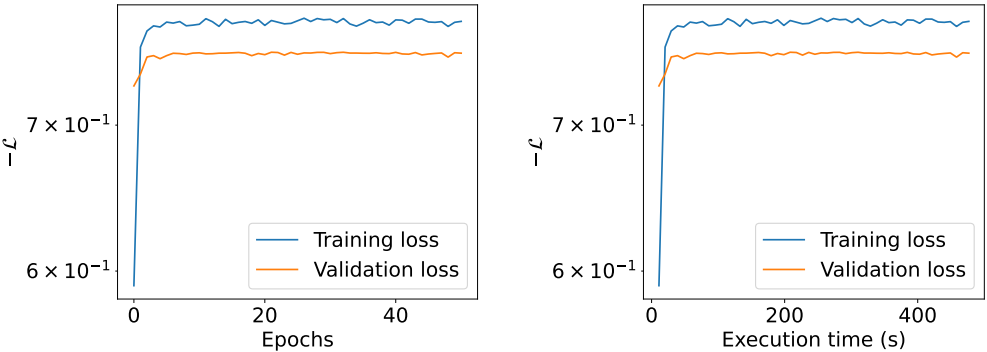


Figure 3: Evolution of the loss function as a function of epochs (left) and of time (right).

Numerical results

Image deblurring

For the numerical experiments, we consider the image deblurring problem with Total Variation regularization, which can be formulated as

$$\underset{u\in\mathbb{R}^d}{\operatorname{argmin}} \underbrace{\frac{1}{2}\|Au-b^\delta\|^2}_{\text{data fidelity}} + \underbrace{\lambda\,\mathrm{TV}(u)}_{\text{regularization}},$$

which is a specific instance of the model in eq. (2) where

- $f(u)=\frac{1}{2}\|Au-b^\delta\|^2$,
- $h(Wu)=\lambda\,\mathrm{TV}(u)$,
- $A\in\mathbb{R}^{d\times d}$ is the blurring operator,
- $\lambda>0$ is a regularization parameter
- $b^\delta\in\mathbb{R}^d$ is the observed image (blurred image with noise).

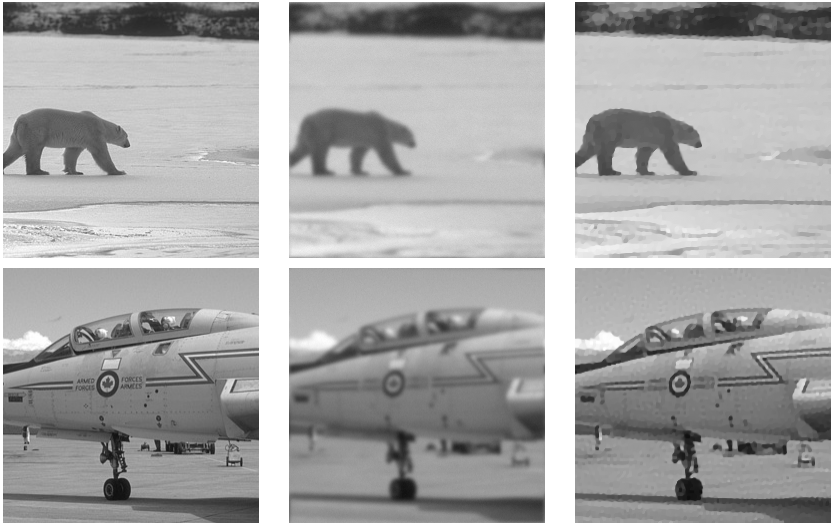
Total Variation

The TV is the Total Variation operator defined as

$$\mathrm{TV}(x) = \sum_{i=1}^{d} \|\nabla_i u\|,$$

where $W \in \mathbb{R}^{2d \times d}$ is the discretization of the gradient operator, and $h : \mathbb{R}^{2d} \to \mathbb{R}$ is the sum of the pixel-wise euclidean norms.

Numerical results



Ground truth | Observed | Reconstructed

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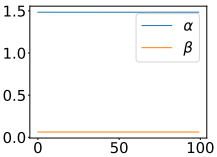
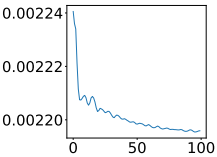
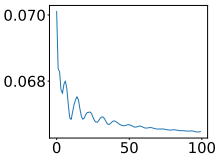
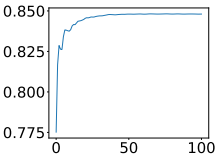
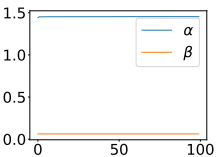
Parameters evolution

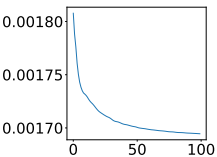


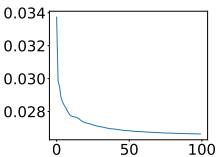


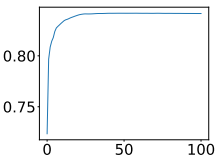












Step sizes (α, β) | Reg. param. (λ) | Prec. param. (ν) | SSIM

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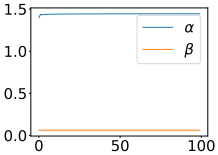

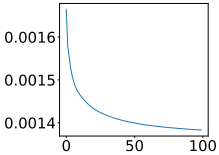

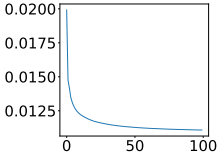

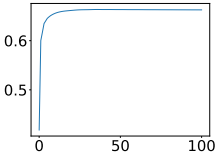

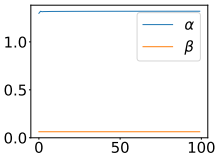

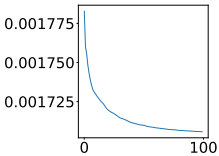

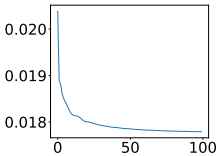

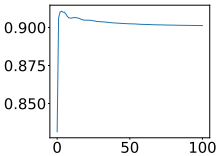

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References

- S. Aleotti, M. Donatelli, R. Krause, G. Scarlato
  A Preconditioned Version of a Nested Primal-Dual Algorithm for Image Deblurring
  J. Sci. Comput. 103, 85 (2025)
- git repository for the PNPD unrolling codes
  `https://github.com/Giuseppe499/PNPD_unrolling`
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Thank you for your attention!